Quantitative Literacy:
Thinking Between the Lines

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Chapter 8:
Voting and Social Choice

Chapter 8: Voting and Social Choice
Lesson Plan

▸ Measuring voting power: Does my vote count?

▸ Voting systems: How do we choose a winner?

▸ Fair division: What is a fair share?

▸ Apportionment: Am I represented?

Learning Objectives:

‣ Understand the variety of ways House representatives can be apportioned

  ‣ Hamilton's solution

  ‣ Jefferson's solution

  ‣ Adjusted divisor methods: Adams and Webster

  ‣ The Huntington-Hill method

‣ **Apportionment:** To find the size of the ideal district for a House representative you use the formula:

$$\text{Ideal district size} = \frac{\text{U. S. popultation}}{\text{House size}}$$

‣ The ideal district size is also known as the **Standard divisor.** This number is used to determine each state's **quota,** or its share of the House of Representatives:

$$\text{State's quota} = \frac{\text{State's population}}{\text{Ideal district size}}$$

▸ **Example:** For the U.S. population of 6,584,255 in 1810, with the House having 181 members:

$$\text{Ideal district size} = \frac{6,584,255}{181} = 36,377.099$$

▸ Image from page 524 here

▸ Each state's population is divided by the ideal district size to calculate its state's quota.

▸ Connecticut had a population of 261,818

$$\text{Quota for Connecticut} = \frac{261,818}{36,377.099} = 7.197$$

8.4 Apportionment: Am I represented?

▸ **Example (cont.):** The rest of the states follow:

State	Population	Quota	Rounded quota
Connecticut	261,818	7.197	7
Delaware	71,004	1.952	2
Georgia	210,346	5.782	6
Kentucky	374,287	10.289	10
Maryland	335,946	9.235	9
Massachusetts	700,745	19.263	19
New Hampshire	214,460	5.895	6
New Jersey	241,222	6.631	7
New York	953,043	26.199	26
North Carolina	487,971	13.414	13
Ohio	230,760	6.344	6
Pennsylvania	809,773	22.261	22
Rhode Island	76,888	2.114	2
South Carolina	336,569	9.252	9
Tennessee	243,913	6.705	7
Vermonth	217,895	5.99	6
Virginia	817,615	22.476	22
Total	**6,584,255**	**181**	**179**

8.4 Apportionment: Am I represented?

▸ A problem arises over what to do with the fractional parts, so methods have been devised to deal with them either by ranking or rounding them somehow.

▸ **Hamilton's solution: Ranking the fractions**

Step 1: Calculate the quota for each state.

Step 2: Give to each state the number of representatives corresponding with the whole number part of the quota.

Step 3: Allocate the leftover House seats, representing the fractions left over, as follows: Rank the states by the size of the fractional part of their quota, from greatest to least, and give one leftover member to each state in that order until the leftovers are exhausted.

State	Quota	Quota rounded down: Initial seats	Fractional part of quota	Added seats	Final seats
Connecticut	7.197	7	0.197		7
Delaware	1.952	1	**0.952**	1	2
Georgia	5.782	5	**0.782**	1	6
Kentucky	10.289	10	0.289		10
Maryland	9.235	9	0.235		9
Massachusetts	19.263	19	0.263		19
New Hampshire	5.895	5	**0.895**	1	6
New Jersey	6.631	6	**0.631**	1	7
New York	26.199	26	0.199		26
North Carolina	13.414	13	**0.414**	1	14
Ohio	6.344	6	0.344		6
Pennsylvania	22.261	22	0.261		22
Rhode Island	2.114	2	0.114		2
South Carolina	9.252	9	0.252		9
Tennessee	6.705	6	**0.705**	1	7
Vermonth	5.99	5	**0.99**	1	6
Virginia	22.476	22	**0.476**	1	23
Total	**181**	**173**		**8**	**181**

▸ **Example:** A city has four districts—North, South, East, and West. There are 10 city councilors allocated according to population with at least one from each district. Given the populations below, calculate the quota for each district and use Hamilton's solution to determine the number of councilors to represent each district:

District	Population
North	3900
East	4800
West	18,000
South	3300
Total	**30,000**

▶ **Solution:** The 10 seats to fill and the population of 30,000 are key.

$$\text{The ideal district size} = \frac{30,000}{10} = 3000$$

District	Quota	Initial seats	Quota remainder	Added seats	Final seats
North	1.3	1	0.3		1
East	1.6	1	0.6	1	2
West	6.0	6	0.0		6
South	1.1	1	0.1		1
Total	**10**	**9**			**10**

▸ **Jefferson's solution: Adjusting the divisor**

Step 1: Start using the ideal district size as the divisor.

Step 2: Calculate the quota for each state by dividing the population by the divisor.

Step 3: Round down each quota to the nearest whole number, but not less than 1, and sum the rounded quotas.

Step 4: If the sum from step 3 is larger than the size of the House, increase the divisor and repeat steps 2 and 3. If the sum is too small, decrease the divisor and repeat steps 2 and 3.

▸ This process is repeated until a divisor is found for which the sum of the rounded quotas is equal to the House size.

‣ **Example:**

State	Population	Quota using divisor 36,377.10	First try: Quota rounded down	Second try: Decrease divisor to 34,000, round down	Third try: Increase divisor to 35,000, round down
Connecticut	261,818	7.197	7	7	7
Delaware	71,004	1.952	1	2	2
Georgia	210,346	5.782	5	6	6
Kentucky	374,287	10.289	10	11	10
Maryland	335,946	9.235	9	9	9
Massachusetts	700,745	19.263	19	20	20
New Hampshire	214,460	5.895	5	6	6
New Jersey	241,222	6.631	6	7	6
New York	953,043	26.199	26	28	27
North Carolina	487,971	13.414	13	14	13
Ohio	230,760	6.344	6	6	6
Pennsylvania	809,773	22.261	22	23	23
Rhode Island	76,888	2.114	2	2	2
South Carolina	336,569	9.252	9	9	9
Tennessee	243,913	6.705	6	7	6
Vermonth	217,895	5.99	5	6	6
Virginia	817,615	22.476	22	24	23
Total	**6,584,255**	**181**	**179**	**187**	**181**

▸ **Example:** Given the populations below, from the previous example, use Jefferson's solution to apportion the 10 councilors to the different districts.

District	Population
North	3900
East	4800
West	18,000
South	3300
Total	**30,000**

▸ **Solution:** 10 councilors and a population of 30,000 gives an ideal district size of 3000. Use this as the first divisor.

▸ **Solution (cont.):** The following table represents Jefferson's solution. At each attempt of apportionment, the quota is rounded down.

District	Population	Quota using divisor 3000	First try: Quota round down	Second try: Decrease divisor to 2500, round down
North	3900	1.3	1	1
East	4800	1.6	1	1
West	18,000	6.0	6	7
South	3300	1.1	1	1
Total	**30, 000**	**10**	**9**	**10**

▸ A desirable trait of any apportionment method is to **stay within quota,** meaning that the final apportionment for each state would be within 1 of the quota.

So the final apportionment should be the quota rounded up one or down one. Apportionment methods that do not stay within the quota are said to **violate quota.**

▸ **The Alabama Paradox:** a problem arising when calculating the quota for certain states.

For a House size of 299, Alabama would be allocated 8 representatives.

For a House size of 300, Alabama receives only 7 representatives.

8.4 Apportionment: Am I represented?

▸ **Example:** Explore the Alabama Paradox:

1. For a House size of 299, determine the ideal district size and verify the quotas for Alabama and Illinois.

2. How do the fractional parts of the quotas for the two states compare for 299 members.

3. How do the fractional parts of the quotas for the two states compare for 300 members?

4. Use parts 2 and 3 to find plausible explanations for the paradox.

8.4 Apportionment: Am I represented?

TABLE 8.2 Alabama Paradox: 1880 Census, Hamilton's Method									
		House size of 299				House size of 300			
State	Population	Quota	Initial seats	Added seats	Total seats	Quota	Initial seats	Added seats	Total seats
Alabama	1,262,505	7.646	7	1	8	7.671	7		7
Arizona	802,525	4.860	4	1	5	4.876	4	1	5
California	864,694	5.237	5		5	5.254	5		5
Colorado	194,327	1.177	1		1	1.181	1		1
Connecticut	622,700	3.771	3	1	4	3.784	3	1	4
Delaware	146,608	0.888	1		1	0.891	1		1
Florida	269,493	1.632	1		1	1.638	1		1
Georgia	1,542,180	9.340	9		9	9.371	9		9
Illinois	3,077,871	18.640	18		18	18.702	18	1	19
Indiana	1,978,301	11.981	11	1	12	12.021	12		12
Iowa	1,624,615	9.839	9	1	10	9.872	9	1	10
Kansas	996,096	6.033	6		6	6.053	6		6
Kentucky	1,648,690	9.985	9	1	10	10.018	10		10
Louisiana	939,946	5.692	5	1	6	5.711	5	1	6
Maine	648,936	3.930	3	1	4	3.943	3	1	4
Maryland	934,943	5.662	5	1	6	5.681	5	1	6
Massachusetts	1,783,085	10.799	10	1	11	10.835	10	1	11
Michigan	1,636,937	9.914	9	1	10	9.947	9	1	10
Minnesota	780,773	4.728	4	1	5	4.744	4	1	5
Mississippi	1,131,597	6.853	6	1	7	6.876	6	1	7
Missouri	2,168,380	13.132	13		13	13.176	13		13
Nebraska	452,402	2.740	2	1	3	2.749	2	1	3
Nevada	62,266	0.377	1		1	0.378	1		1
New Hampshire	346,991	2.101	2		2	2.108	2		2
New Jersey	1,131,116	6.850	6	1	7	6.873	6	1	7
New York	5,082,871	30.783	30	1	31	30.886	30	1	31
North Carolina	1,399,750	8.477	8		8	8.505	8		8
Ohio	3,198,062	19.368	19		19	19.433	19		19
Oregon	174,768	1.058	1		1	1.062	1		1
Pennsylvania	4,282,891	25.938	25	1	26	26.025	26		26
Rhode Island	276,531	1.675	1	1	2	1.680	1	1	2
South Carolina	995,577	6.029	6		6	6.050	6		6
Tennessee	1,542,359	9.341	9		9	9.372	9		9
Texas	1,591,749	9.640	9		9	9.672	9	1	10
Vermont	332,286	2.012	2		2	2.019	2		2
Virginia	1,512,565	9.160	9		9	9.191	9		9
West Virginia	618,457	3.745	3	1	4	3.758	3	1	4
Wisconsin	1,315,497	7.967	7	1	8	7.993	7	1	8
Total	49,371,340	299	279	20	299	300	282	18	300

▸ **Solution:**

1. The table shows the total population is 49,371,340.

   ▸ For a house size of 299, the ideal district size is total population divided by 299, yielding about 165,121.539.

   ▸ To verify the quotas divide the state population by the ideal district size.

   ▸ For Alabama, the quota is $\dfrac{1,262,505}{165,121.539} = 7.646$

   ▸ For Illinois, the quota is $\dfrac{3,077,871}{165,121.539} = 18.640$

▸ **Solution (cont.):**

2. With 299 members, Alabama's fractional part is 0.646 and Illinois's is 0.640. Alabama has the larger fractional part.

3. The table shows that with 300 members, the fractional part for Alabama is 0.671 and Illinois's is 0.702. Illinois has the larger fractional part.

4. From parts 2 and 3, it is apparent that increasing the number of seats changes the fractional parts of the quotas for the two states. Because Hamilton's method allocates leftover seats according to higher fractional parts, it is reasonable that Alabama could lose a seat and Illinois could gain a seat with the change.

‣ Other problems arising from Hamilton's procedure are the:

  ‣ **Population paradox:** Where the numbers in question are being affected by different state populations growing at different rates, **and**

  ‣ **New states paradox:** Where problems arise because of new states joining the United States.

‣ Methods were devised to address some of the problems that arose with the procedures, for example the three adjusted divisor methods for apportionment.

▸ **Three Adjusted Divisor Methods for Apportionment:**

1. Calculate the ideal district size by dividing the total population by the size of the House.

2. Calculate the quota for each state by dividing its population by the divisor.

3. Round each quota to a whole number as follows:

   ▸ **Jefferson's Method:** Round down, but not less than 1

   ▸ **Adam's Method:** Round up.

   ▸ **Webster's Method:** Round to the nearest whole number, up if the fractional part is 0.5 or greater and down otherwise, but not less than 1.

   Then sum the resulting rounded quotas.

4. If the sum from step 3 is larger than the size of the House, increase the divisor and repeat steps 2 and 3. If the sum is too small, decrease the divisor and repeat steps 2 and 3.

   Continue this process until a divisor is found for which the sum of the rounded quota is equal to the number of House members.

8.4 Apportionment: Am I represented?

▸ **Example:** According to the 2010 census, the population was:

State	Population
Montana	994,416
Alaska	721,523
Florida	18,900,773
West Virginia	1,859,815
Total	**22,476,527**

Congress allocated 32 seats in total to these states.

▸ **Example (cont.):**

1. Calculate the ideal district size and each state's quota. Keep three places beyond the decimal point.

2. Calculate the apportionment of the 32 seats according to Hamilton's method.

3. Calculate the apportionment of the 32 seats according to Jefferson's method.

4. Calculate the apportionment of the 32 seats according to Adam's method.

5. Calculate the apportionment of the 32 seats according to Webster's method.

▸ **Solution:**

1. The ideal district size $= \frac{22{,}476{,}527}{32} = 702{,}391.469$.

   The quota for each state is calculated by dividing the state's population by this divisor.

State	Population	Quota
Montana	994,416	1.416
Alaska	721,523	1.027
Florida	18,900,773	26.909
West Virginia	1,859,815	2.648
Total	**22, 476, 527**	**32**

▸ **Solution (cont.):**

2. For Hamilton's method, round the quotas down. This fills 30 seats. The remaining two are given to the largest fractional parts of the states' quotas, Florida and West Virginia.

State	Population	Quota	Quota round down	Fractional part of quota	Added seats	**Final seats**
Montana	994,416	1.416	1	0.416	0	**1**
Alaska	721,523	1.027	1	0.027	0	**1**
Florida	18,900,773	26.909	26	0.909	1	**27**
W. Virginia	1,859,815	2.648	2	0.648	1	**3**
Total	**22,476,527**	**32**	**30**		**2**	**32**

▸ **Solution (cont.):**

3. For Jefferson's method, the quota is again rounded down; the total is less than 32 so the divisor is adjusted to a smaller number of 700,000. On this attempt it is still too few, so a third divisor of 675,000 is used.

State	Population	Quota using divisor 702,391.469	Frist try Quota round down	Second try 700,000: round down	Third try 675,000: round down
Montana	994,416	1.416	1	1	1
Alaska	721,523	1.027	1	1	1
Florida	18,900,773	26.909	26	27	28
W. Virginia	1,859,815	2.648	2	2	2
Total	**22,476,527**	**32**	**30**	**31**	**32**

▸ **Solution (cont.):**

4.   For Adam's method, the initial allocation is made by rounding each quota up. The total is more than 32 so the divisor is adjusted higher to 750,000. The 32 desired seats are obtained.

State	Population	Quota using divisor 702,391.469	First try: round up	Divisor of 750,000: round up
Montana	994,416	1.416	2	2
Alaska	721,523	1.027	2	1
Florida	18,900,773	26.909	27	26
W. Virginia	1,859,815	2.648	3	3
Total	**22,476,527**	**32**	**34**	**32**

▸ **Solution (cont.):**

5. For Webster's Method, round the initial quotas to the nearest whole number, round down if the decimal is less than 0.5 and up otherwise. Because the total from the initial allocations is 32, no adjustments need to be made.

State	Population	Quota	First try: round about	Total seats
Montana	994,416	1.416	1	1
Alaska	721,523	1.027	1	1
Florida	18,900,773	26.909	27	27
W. Virginia	1,859,815	2.648	3	3
Total	**22,476,527**	**32**	**32**	**32**

▸ The **Huntington-Hill Method:** is an adjusted divisor method and it is the one that is used today. It follows the steps of the other methods, but the divisor is found using a **geometric mean**: if n is the whole number part of the quotient, to find the geometric mean use the formula:

$$\boxed{\text{Geometric mean } = \sqrt{n(n+1)}}$$

The quota for each state is rounded up if it is at least as large as the geometric mean or down otherwise.

A table of Geometric means follows:

▸ **Huntington-Hill method (cont.):**

TABLE 8.3 Geometric Means

Whole number	Geometric mean	Whole number	Geometric mean	Whole number	Geometric mean
1	1.414	11	11.489	21	21.494
2	2.449	12	12.490	22	22.494
3	3.464	13	13.491	23	23.495
4	4.472	14	14.491	24	24.495
5	5.477	15	15.492	25	25.495
6	6.481	16	16.492	26	26.495
7	7.483	17	17.493	27	27.495
8	8.485	18	18.493	28	28.496
9	9.487	19	19.494	29	29.496
10	10.488	20	20.494	30	30.496

▸ **Example:** Use the Huntington-Hill method to allocate 32 seats to the four states from previous example.

▸ **Solution:**

State	Population	Quota: Divisor of 702,391.469	Geometric mean	1st try: Round quota	2nd try: Divisor 640,000	Rounded quota
Montana	994,416	1.416	1.414	2	1.401	1
Alaska	721,523	1.027	1.414	1	1.016	1
Florida	18,900,773	26.909	26.495	27	26.621	27
W. Virginia	1,859,815	2.648	2.449	3	2.619	3
Total	**22,476,527**	**32**		**33**		**32**

8.4 Apportionment: Am I represented?
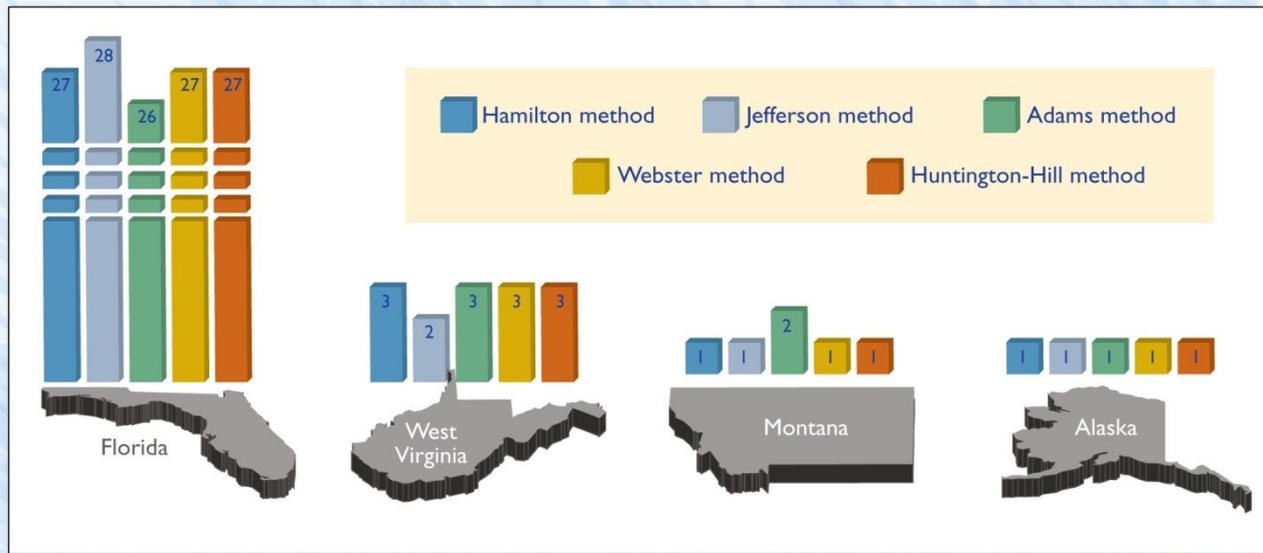
FIGURE 8.1 Results of various apportionment methods.

▸ **Voting power:** does your vote count

- ▸ coalitions, quotas, swing voters
- ▸ Banzhaf power index
- ▸ Stanley-Shubik power index

▸ **Voting systems:** choosing a winner

- ▸ plurality
- ▸ top-two runoff
- ▸ elimination runoff
- ▸ Borda count
- ▸ Condorcet winner

Fair division:

- Methods: divide and choose

  adjusted winner procedure

Apportionment:

- Hamilton's method
- Jefferson's method
- Adjusted divisor methods
- Huntington-Hill method